Indian Statistical Institute
Semestral Examination
Differential Topology-MMath II

Answer all questions.
Max Marks: 60 Time: 3 hours

(1) (a) Show that the interval $[-1,1]$ is not diffeomorphic to
$X = \{(x,y) \mid (0 \leq x \leq 1 \text{ and } y = 0) \text{ or } (x = 1 \text{ and } 0 \leq y \leq 1)\}$.
(b) Consider the map $f : S^2 \longrightarrow S^1 \times S^1$ defined by
$$f(x,y,t) = ((x, \sqrt{y^2+t^2}), e^{2\pi it}).$$
Does f have any critical point?.
(c) Does there exist a submersion $g : S^1 \times S^1 \longrightarrow \mathbb{RP}^2$?
(d) Show that the vector field $\partial/\partial x$ on $\mathbb{R}^2 - 0$ is not complete.
(e) Show that the tangent bundle TS^1 of S^1 is diffeomorphic to $S^1 \times \mathbb{R}$. [4 x 5 = 20]

(2) (a) Show that the real projective space $\mathbb{RP}^n$ is a smooth manifold. Give a complete proof. [8]
(b) Show that S^n is orientable for all $n \geq 1$ by exhibiting a nowhere zero n-form. Show that $\mathbb{RP}^n$ is orientable if n is odd. [8]
(c) Let $f : M \longrightarrow N$ be a smooth map. Smooth vector fields X on M and Y on N are said to be f-related if $df \circ X = Y \circ f$. Suppose that X, X_1 are smooth vector fields on M and Y, Y_1 smooth vector fields on N. (i) If X is f-related to Y and X_1 is f-related to Y_1 show that $[X, X_1]$ is f-related to $[Y, Y_1]$. (ii) Suppose that $df(X(p)) = df(X(q))$ whenever $f(p) = f(q)$. Is there a smooth vector field Z on N that is f-related to X? [8]

(3) (a) State the stability theorem. Show by examples that the properties (i) being an embedding, (ii) being transverse to a given submanifold are not stable properties of maps defined on non-compact domains. [2+2+2=6]
(b) Compute $H^i_{dR}(S^1)$, $i \geq 0$. [10]